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THE MATRIX ALGEBRA FOR IMPLICATIONS

TWO developments of the "calculus of propositions,"—or algebra of implications,—may be said to give the same system provided the same propositions appear in each. Postulates of one development may be theorems of the other, and the same system may result from any number of different sets of assumptions. The system of material implication results from any set of postulates for the Boolean algebra, when the variables are interpreted as propositions.[1] The meaning of implication in this system is such that "p implies q" is not equivalent to "q can be inferred from p." In previous papers,[2] the writer has given postulates for another system, "Strict Implication," in which "p implies q" means exactly "q can be inferred from p."

The algebra to be presented in this paper is a new system, more comprehensive than any previously developed. The system of material implication, the system of strict implication, and at least one other system,—a calculus of consistencies which has never been developed,—are all included in it. We shall call it the "Matrix Algebra for Implications."[3] A set of assumptions for this system follows.

PRIMITIVE IDEAS

The primitive ideas are the same as a possible set for material implication, with one additional idea,—impossibility (the truth value of a proposition which implies its own denial).

Propositions. p, q, r, *etc.*, symbolize propositions or propositional functions. (A propositional function is an expression, involving a variable or variables, which becomes a proposition when a value of the variable or variables is assigned.)

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[1] Postulates for this system have been given by Huntington, Müller (after Schröder), Peano, in Whitehead's "Universal Algebra," and in "Principia Mathematica," Whitehead and Russell. For the most economical set, see H. M. Sheffer, *Trans. Am. Math. Soc.*, Vol. XIV., pages 481–488.

[2] "A New Algebra of Implications, etc.," this JOURNAL, Vol. X., page 428, and "The Calculus of Strict Implications," *Mind*, N. S., No. 90.

[3] This designation is not intended to imply that this algebra is the most universal possible. Still more comprehensive systems for implication may be discovered.

Negation. $-p$ symbolizes "not $-p$" or "p is false."

Impossibility. $\sim p$ symbolizes "p is impossible" or "It is impossible that p be true." The intent of this idea will become clearer as the system is developed.

Product. pq symbolizes "p and q both" or "p is true and q is true."

Equivalence. $p = q$ means "p is equivalent to q." This is the defining relation.

TRUTH VALUES

Systems previously developed have only two truth values,—truth and falsity. The addition of the idea "impossibility" gives us five truth values, all of which are familiar logical ideas.

1. $\sim p$. p is impossible; it is impossible that p be true.

2. $-p$. p is false.

3. $-\sim p$. p is possible; it is possible that p be true. Strictly, this should be written $-(\sim p)$. The parenthesis is regularly omitted for typographical reasons.

4. p. p is true.

5. $\sim -p$. p is necessary; it is impossible that p be false. The sign $\sim$ affects the whole of $-p$: parenthesis is regularly omitted, as in 3.[4]

The reader need be at no pains to grasp $-\sim p$ and $\sim -p$ as simple ideas. It is sufficient to understand $-p$ and $\sim p$ and to remember that each prefix affects the letter as already modified by those nearer it.

DEFINITIONS

The dyadic relations of propositions, with which the algebra principally deals, can be defined as truth values of products.

1. *Consistency.* $(p \circ q) = -\sim (pq)$. $\sim (pq)$, "It is impossible that p and q both be true," means "p and q are inconsistent." Hence its negative, $-\sim (pq)$, represents "p is consistent with q."

2. *Implication (inference).* $(p \supset q) = \sim (p-q)$.[5] "$p$ implies q

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[4] It might be remarked that symmetry demands a sixth truth value. $-\sim p$ symbolizes strictly, not "p is possible," but "p is possibly true." We should, then, have "p is possibly false." In fact we do have it. $-\sim -p$ (each prefix affecting the whole of what follows it) is "It is false that it is impossible that p be false," *i. e.*, "p is possibly false." We may equally well observe that when "impossible" is distinguished from merely "false,"—as it is in ordinary logical thinking,—there are an indefinite, perhaps infinite, number of truth values. $-(-p)$ is equivalent to p, as will be postulated, but $\sim\sim p$ is irreducible, as is $-\sim -p$, $\sim -\sim p$, $\sim\sim -p$, *etc.*

[5] We make use of symbols which have figured in "Principia Mathematica" and elsewhere *with different meanings.* The excuse for this is the availability of the types in question.

(q can be inferred from p)" is here defined to mean "It is impossible that p be true and q false."

3. *Dilemmatic Disjunction.* $(p \vee q) = \sim(-p-q)$.

4. *Material Implication.* $(p < q) = -(p-q)$. $(p < q)$ means: "It is false that p is true and q false." This is the implication relation of all developments of material implication. $(p < q)$ does *not* mean "q can be inferred from p." When we speak of implication hereafter, the relation defined in 2 is always intended. When $(p < q)$ is meant, we shall write "material implication" and "materially implies."

5. *Non-dilemmatic Disjunction.* $(p+q) = -(-p-q)$. The difference between $(p+q)$ and $(p \vee q)$ is exactly analogous to that between $(p < p)$ and $(p \supset q)$. The distinction can not easily be preserved in English: both $(p \vee q)$ and $(p+q)$ would be rendered, "Either p or q." It is probably the confusion of these two ideas which originally led to the development of material implication.[6]

The first three relations, $(p \circ q)$, $(p \supset q)$, and $(p \vee q)$, involve the idea of impossibility and do not belong to material implication. (pq), $(p < q)$, and $(p+q)$ do belong to material implication and do not involve the idea of impossibility. An interesting analogy between these two sets will appear shortly.

For the sake of completeness we add two definitions.

6. $(p \equiv q) = [(p < q)(q < p)]$.

7. $(p = q) = [(p \supset q)(q \supset p)]$. $(p \equiv q)$ is "material equivalence" (always so referred to in this paper), which is equivalence of truth value, not of intension. $(p = q)$, the defining relation, is itself defined, although a primitive idea, because this definition enables us to deduce other definitions. Obviously the analogy above referred to extends to $(p = q)$ and $(p \equiv q)$.

POSTULATES

P1. $(pq) \supset (qp)$.
P2. $(qp) \supset p$.
P3. $p \supset (pp)$.
P4. $[p(qr)] \supset [q(pr)]$.
P5. $[p \supset (q \supset r)] \supset [q \supset (p \supset r)]$.
P6. $(p \supset q) \supset [(q \supset r) \supset (p \supset r)]$.
P7. $p = -(-p)$.
P8. $(p \supset q) = (\sim q \supset \sim p)$.
P9. $\sim p \supset -p$.

In this set of postulates, economy and logical neatness have been somewhat sacrificed to clearness. If $(p+q)$ and the idea of necessary truth, $\propto p$, were taken as primitives, in place of (pq) and $\sim p$,

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[6] See "Implication and the Algebra of Logic," *Mind*, N. S., No. 84.

P7 could be proved, P8 reduced to an implication, and P4 and P5 replaced by a single postulate. But the ideas of $(p+q)$ and $\propto p$ are much harder to keep in mind than (pq) and $\sim p$. In the set as given, P7 might be reduced to $p \supset -(-p)$, but this would require several long and difficult proofs which we are able to omit with P7 as written.

P8 is equivalent to the pair $(p \supset q) \supset (-\sim p \supset -\sim q)$ and $(-p \supset -q) \supset (\sim p \supset \sim q)$. These propositions are more "self-evident" than the postulate, but express exactly the same relation.

OPERATIONS

Substitution.—Any proposition or propositional function may be substituted for *p, q, r, etc.,* in any primitive proposition or any theorem. Also, expressions which are equivalent,—$(p = q)$, not $(p \equiv q)$,—may be substituted for one another.

Inference.—If p is asserted and $(p \supset q)$ is asserted, q may be asserted.

Production.—If p and q are separately asserted, (pq) may be asserted. (This operation is not indispensable, but occasionally renders proof much less cumbersome.)

The first theorem will be proved in full. Later proofs will be abbreviated or indicated only.

Theorem 1. $(pq) \supset p$.

P6. $\{(pq)/p;\ (qp)/q;\ p/r\}$: $(\text{P1}) \supset \{(\text{P2}) \supset [(pq) \supset p]$.

This proof may be read: "Postulate 6; when (pq) is substituted for p; (qp) for q; and p for r; states that: Postulate 1 implies that Postulate 2 implies $(pq) \supset p$." Let us make the substitutions indicated in Postulate 6. We then have

$$[(pq) \supset (qp)] \supset \{[(qp) \supset p] \supset [(pq) \supset p].$$

The expression in the first pair of brackets is P1. Since P1 is asserted, its implication, which follows it, may be asserted. The expression in the second pair of brackets is P2. Since P2 is asserted, its consequence, Theorem 1, may be asserted.

Th. 2. $(p \supset q) \supset (\sim q \supset \sim p)$.

By Def. 7,

$(\text{P8}) = \{[(p \supset q) \supset (\sim q \supset \sim p)], [(\sim q \supset \sim p) \supset (p \supset q)]\}$ (a)

Th. 1; (a) $\supset$ Q. E. D.

We introduce here abbreviations of proof as follows: (a), or (b), etc., is placed after a lemma which has been established, and thereafter in the same proof we write (a), or (b), etc., instead of that lemma. Also, we shall frequently write "Q. E. D." in the last line of proof in place of the theorem to be proved. In the second line of this proof, the substitutions which it is necessary to make in Th. 1 in order to get $(a) \supset [(p \supset q) \supset (\sim q \supset \sim p)]$ are not stated because they are obvious. If (a) is pq, p is clearly Th. 2.

Theorem 2 is one of the implications contained in P8. By Def. 7, a pair of implications may be substituted for any equivalence. By P2 and Th. 1, either of these implications may be taken separately.

Th. 3. $(-p \supset q) \supset (-q \supset p)$.
P1 $\{-q/p;\ -p/q\}$: $(-q-p) \supset (-p-q)$ (a)
Th. 2 $\{(-q-p)/p;\ (-p-q)/q\}$: $(a) \supset [\sim(-p-q) \supset \sim(-q-p)$
by Def. 2, $(-p \supset q) \supset (-q \supset p)$.

Th. 4. $(p \supset q) \supset (-q \supset -p)$.
Th. 3 $\{-p/p\}$: $[-(-p) \supset q] \supset [(-q \supset -p)]$.
by P7, $(p \supset q) \supset (-q \supset -p)$.

Th. 5. $(p \supset -q) \supset (q \supset -p)$.

Similar proof.

Th. 6. $(-p \supset -q) \supset (q \supset p)$.

Similar proof.

Theorems 3, 4, 5, and 6 are the four forms of the principle of permutation. Use of any one of these theorems is indicated hereafter by "perm."

Th. 7. $(p \circ q) \supset (q \circ p)$.
Th. 2 $\{(pq)/p;\ (qp)/q\}$: $(P1) \supset [\sim(qp) \supset \sim(pq)]$ (a).
Perm. $(a) \supset [-\sim(pq) \supset -\sim(qp)]$
by Def. 1, $(p \circ q) \supset (q \circ p)$.

Th. 8. $(q \circ p) \supset -\sim p$.
Th. 2 $\{(qp)/p;\ p/q\}$: $(P2) \supset [\sim p \supset \sim(qp)]$ (a).
Perm, $(a) \supset [-\sim(qp) \supset -\sim p]$
by Def. 1, $(q \circ p) \supset -\sim p$.

Th. 9. $-\sim p \supset (p \circ p)$.

Similar proof, using P3.

Th. 10. $[p \circ (qr)] \supset [q \circ (pr)]$.

Similar proof, using P4.

Th. 11. $[p \circ (q \circ r)] \supset [q \circ (p \circ r)]$.
P5 $\{q/p;\ p/q;\ -r/r\}$: $[q \supset (p \supset -r)] \supset [p \supset (q \supset -r)]$
by Def. 2, $\{q \supset \sim[p-(-r)]\} \supset \{p \supset \sim[q-(-r)]\}$
by P7, $[q \supset \sim(pr)] \supset [p \supset \sim(qr)]$
by Def. 2, $\sim[q-\sim(pr)] \supset \sim[p-\sim(qr)]$ (a).
Perm. $(a) \supset \{-\sim[p-\sim(qr)] \supset -\sim[q-\sim(pr)]$
by Def. 1, $[p \circ (q \circ r)] \supset [q \circ (p \circ r)]$.

We can now exhibit the analogy between products and consistencies, facts and possibilities, which runs all through the system.

P1.	$(pq) \supset (qp)$.	Th. 7.	$(p \circ q) \supset (q \circ p)$.
P2.	$(qp) \supset p$.	Th. 8.	$(q \circ p) \supset - \sim p$.
P3.	$p \supset (pp)$.	Th. 9.	$- \sim p \supset (p \circ p)$.
P4.	$[p(qr)] \supset [q(pr)]$.	Th. 11.	$[p \circ (q \circ r)] \supset [q \circ (p \circ r)]$.[7]

The same analogy holds between non-dilemmatic and dilemmatic disjunctions, as follows:

$(p + q) \supset (q + p)$	$(p \vee q) \supset (q \vee p)$
$p \supset (p + q)$	$\sim -p \supset (p \vee q)$
$(p + p) \supset p$	$(p \vee p) \supset \sim -p$
$[p + (q + r)] \supset [q + (p + r)]$	$[p \vee (q \vee r)] \supset [q \vee (q \vee r)]$

We prove the two theorems given in the first line to illustrate the method by which theorems involving disjunctions are deduced.

Th. 12. $(p + q) \supset (q + p)$.
P1 $\{-q/p;\ -p/q\}$: $(-q-p) \supset (-p-q)$ (a).
Perm. (a) $\supset [-(-p-q) \supset -(-q-p)$
by Def. 5, $(p + q) \supset (q + p)$.

Th. 13. $(p \vee q) \supset (q \vee p)$.
P1 $\{-q/p;\ -p/q\}$: $(-q-p) \supset (-p-q)$ (a).
P9. (a) $\supset [\sim(-p-q) \supset \sim(-q-p)$
by Def. 3, $(p \vee q) \supset (q \vee p)$.

Again, the same analogy holds between material implications and implications of the type of inference:

$(-p < q) \supset (-q < p)$	Th. 2. $(-p \supset q) \supset (-q \supset p)$
$-p \supset (p < q)$	$\sim p \supset (p \supset q)$
$(p < -p) \supset -p$	$(p \supset -p) \supset \sim p$
$p \supset (q < p)$	$\sim -p \supset (q \supset p)$
$(-p < p) \supset p$	$(-p \supset p) \supset \sim -p$

Note that the *main* implication in theorems in both columns is of the type of inference.

We prove the second theorem in each column.

Th. 14. $-p \supset (p < q)$.
Th. 1 $\{-q/q\}$: $(p-q) \supset p$ (a).
Perm. (a) $\supset [-p \supset -(p-q)$
by Def. 4, $-p \supset (p < q)$.

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[7] If P4 could be written $(pqr) \supset (qpr)$, an exact analogue $(p \circ q \circ r) \supset (q \circ p \circ r)$, could be derived from it. As it is, P4 proves Th. 10, and Th. 11 is derived from P5.

Th. 15. $\sim p \supset (p \supset q)$.
Th. 1 $\{-q/q\}$: $(p - q) \supset p$ (a).
Th. 2. (a) $\supset [\sim p \supset \sim (p - q)$
by Def. 2, $\sim p \supset (p \supset q)$.

The Calculus of Consistencies

This persistent analogy between the relations which figure in material implication, pq, $p + q$, and $p < q$, and the set involving the idea of impossibility, $p \circ q$, $p \vee q$, and $p \supset q$, suggests that there may be a calculus of possibilities and impossibilities, similar to the system of material implication. We give below assumptions sufficient for such a system, which might be called the calculus of consistencies.

Primitive Ideas.—Propositions, negation, equivalence, and consistency,—all symbolized as before.

Definitions:

$$(a) \quad (p \vee q) = -(-p \circ -q)$$
$$(b) \quad (p \supset q) = (-p \vee q)$$

Postulates.—These postulates will be lettered, *a*, *b*, . . . to avoid confusion with the postulates of the matrix algebra. For economy of assumption, they will be expressed in terms of the relation $\vee$. But since this relation is easily confused with $+$, we give a', b', . . ., equivalent respectively to *a*, *b*. . . .

(*a*) $(p \vee q) \supset (q \vee p)$. (*a′*) $(p \circ q) \supset (q \circ p)$.
(*b*) $(p \vee p) \supset (p \vee q)$. (*b′*) $(p \circ q) \supset (p \circ p)$.
(*c*) $(p \vee p) \supset p$. (*c′*) $p \supset (p \circ p)$.
(*d*) $[p \vee (q \vee r)] \supset [q \vee (p \vee r)]$. (*d′*) $[p \circ (q \circ r)] \supset [q \circ (p \circ r)]$.
(*e*) $(q \supset r) \supset [(p \vee q) \supset (p \vee r)]$. (*e′*) $(q \supset r) \supset [(p \circ q) \supset (p \circ r)]$.

The force of *b* and *c* may not at once be clear. *B*, or b', expresses the assumption that if *p* is consistent with some (any) other proposition, *q*, it is self-consistent or non-contradictory. This is equivalent to $-(p \circ p) \supset -(p \circ q)$, "A proposition not self-consistent is inconsistent with any other," and to $(-p \supset p) \supset (q \supset p)$, "A proposition implied by its own negation is implied by any (every) proposition." *C*, or c', states that every true proposition is self-consistent.

This calculus has never been developed. It is not identical with Mrs. Ladd-Franklin's algebra[8] based on the notions of consistency and inconsistency, but has interesting relations to that system. All the propositions of the calculus of consistencies can be proved from the assumptions of the matrix algebra for implications.

[8] In "Studies in Logic by Members of Johns Hopkins University," ed., Pierce.

MATERIAL IMPLICATION

The propositions of the system of material implication also can be proved from the assumptions of the matrix algebra. We proceed to deduce the primitive propositions for material implication given in "Principia Mathematica."[9]

The primitive ideas; p, q, r, etc.; $-p$; and $(p = q)$; belong to material implication as to the matrix algebra. The other primitive is "either p is true or q is true, where the alternatives are to be not mutually exclusive" (p. 97). The use of this relation shows it to be non-dilemmatic disjunction, which we symbolize, $p + q$.

Of the definitions, one, $(p \equiv q) = [(p < q)(q < p)]$, is already assumed in the matrix algebra. The other two may be deduced. (Primitive propositions and theorems used in the proofs are, of course, those of the matrix algebra.)

Th. 16. $(p < q) = (-p + q)$[10]

Def. 4. $(p < q) = -(p - q)$.

Def. 5. $\{-p/p\}$: $(-p + q) = -[-(-p) - q)$

by P7, $(-p + q) = -(p - q)$

by substitution, $(p < q) = (-p + q)$.

For the deduction of the next definition, three lemmas must first be proved.

Th. 17. $(p = q) \supset (p \supset q)$.

Def. 7 $\{(p = q)/p;\ [(p \supset q)(q \supset p)]/q\}$:

$(\text{Def. } 7) = [\{(p = q) \supset [(p \supset q)(q \supset p)]\}$

$\{[(p \supset q)(q \supset p)] \supset (p = q)\}]$ (a).

(The last three lines should be read as one.)

Th. 1. $(a) \supset \{(p = q) \supset [(p \supset q)(q \supset p)]$ (b).

Th. 1. $[(p \supset q)(q \supset p)] \supset (p \supset q)$ (c).

P6. $(b) \supset [(c) \supset$ Q. E. D.

Th. 18. $(p = q) \supset (q \supset p)$.

Similar proof.

Th. 19. $[(p \supset q)(q \supset p)] \supset (p = q)$.

P2. [(a) in proof of Th. 17] $\supset$ Q. E. D.

Th. 20. $(pq) = -(-p + -q)$ [*Prin. Math.*, *3.01].

Def. 5 $\{-p/p;\ -q/q\}$: $(-p + -q) = -[-(-p) - (-q)]$

by P7, $(-p + -q) = -(pq)$ (a).

Th. 17. $(a) \supset [(-p + -q) \supset -(pq)]$ (b).

Th. 18. $(a) \supset [-(pq) \supset (-p + -q)]$ (c).

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[9] Pp. 95–101, 114, and 120. We neglect those refinements of the development given in "Principia Mathematica" which are due to the theory of types.

[10] "Prin. Math.," *1.01.

Perm. $(b) \supset [(pq) \supset -(-p + -q)]$ (d).
Perm. $(c) \supset [-(-p + -q) \supset (pq)]$ (e).
by production of (d) and (e),
$\{[(pq) \supset -(-p + -q)][-(-p + -q) \supset (pq)]\}$ (f).
Th. 19. $(f) \supset$ Q. E. D.

We proceed to the proof of the postulates as given in "Principia Mathematica." The first step is the derivation of theorems similar to these postulates, but not identical with them.

Th. 21. $(p + p) \supset p$.
P3 $\{-p/p\}$: $-p \supset (-p-p)$ (a).
Perm. $(a) \supset [-(-p-p) \supset p$
by Def. 5, $(p + p) \supset p$.

Th. 22. $q \supset (p + q)$.
P2 $\{-p/q;\ -q/p\}$: $(-p-q) \supset -q$ (a).
Perm. $(a) \supset [q \supset -(-p-q)$
by Def. 5, $q \supset (p + q)$.

Th. 23. $(p + q) \supset (q + p)$.

Similar proof, using P1.

Th. 24. $[p + (q + r)] \supset [q + (p + r)]$.
P4 $\{-q/p;\ -p/q;\ -r/r\}$: $[-q(-p-r)] \supset [-p(-q-r)]$ (a).
Perm. $(a) \supset \{-[-p(-q-r)] \supset -[-q(-p-r)]$
by P7, $-\{-p-[-(-q-r)]\} \supset -\{-q-[-(-p-r)]\}$
by Def. 5, $[p + -(-q-r)] \supset [q + -(-p-r)]$
by Def. 5, $[p + (q + r)] \supset [q + (p + r)]$.

Th. 25. $(\sim p \supset \sim q) \supset (q \supset p)$.
Th. 18. $(P8) \supset$ Q. E. D.

Th. 26. $(\sim p \supset \sim q) \supset (-p \supset -q)$.
Perm. $(q \supset p) \supset (-p \supset -q)$ (a).
P6. $(a) \supset [($Th. 25$) \supset$ Q. E. D.

Th. 27. $[(p \supset q) \supset (r \supset s)] \supset [(p < q) \supset (r < s)$.
Th. 26 $\{(p-q)/p;\ (r-s)/q\}$:
$[\sim(p-q) \supset \sim(r-s)] \supset [-(p-q) \supset -(r-s)]$
by Defs. 2 and 4, Q. E. D.

Th. 28. $(p \supset q) \supset (p < q)$.
P9 $\{(p-q)/p\}$: $\sim(p-q) \supset -(p-q)$
by Defs. 2 and 4, Q. E. D.

Th. 29. $(q \supset r) \supset [(p < q) \supset (p < r)]$.
P5. $(P6) \supset \{(q \supset r) \supset [(p \supset q) \supset (p \supset r)]\}$ (a).
P6. $(a) \supset [($Th. 25$) \supset$ Q. E. D.

Th. 30. $(q \supset r) \boldsymbol{\supset} [(p + q) \boldsymbol{\supset} (p + r)]$.

Th. 29 $\{-p/p\}$: $(q \supset r) \boldsymbol{\supset} [(-p < q) \boldsymbol{\supset} (-p < r)]$

by Def. 4, $(q \supset r) \boldsymbol{\supset} [-(-p-q) \boldsymbol{\supset} - (-p-r)]$

by Def. 5, Q. E. D.

Th. 31. $(q < r) \boldsymbol{\supset} [(p + q) < (p + r)]$.

Th. 27 $\{q/p;\ r/q;\ (p+q)/r;\ (p+r)/s\}$: (Th. 30) $\boldsymbol{\supset}$ Q.E.D.

We now have a set of theorems which differ from the postulates for material implication in only one respect,—each of these theorems contains one implication of the type $\boldsymbol{\supset}$. But Th. 28 tells us that from any implication, $(p \supset q)$, the corresponding material implication, $(p < q)$, can be inferred. By use of this theorem we prove immediately:

Th. 32. $(p + p) < p$[11] (from Th. 21).

Th. 33. $q < (p + q)$[12] (from Th. 22).

Th. 34. $(p + q) < (q + p)$[13] (from Th. 23).

Th. 35. $[p + (q + r)] < [q + (p + r)]$[14] (from Th. 24).

Th. 36. $(q < r) < [(p + q) < (p + r)]$[15] (from Th. 31).

These are the postulates for material implication. However, the system of material implication, as previously developed, requires an operation which may be stated: "If p is asserted and $(p < q)$ is asserted, q may be asserted."[16] We do not assume this operation, because $(p < q)$ is not equivalent to "q can validly be inferred from p." But we can prove all the theorems of material implication from the postulates of the matrix algebra in much the same way as we have just proved the postulates. Also, by using the postulates and theorems of the matrix algebra,—*e. g.*, P6 and Th. 27,—as principles of inference, we can prove that the theorems of material implication *can be inferred from* the postulates of material implication. This has never before been shown. Previous developments of material implication have proceeded by means of numerous "mathematical operations" or have proved only that the theorems are *materially implied by* the postulates.

If, however, the opinion expressed in this paper, that the relation of material implication is not equivalent to valid inference, be correct, then the system of material implication has no value as an organon of proof, and its interest is chiefly mathematical and historical.

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[11] "Prin. Math.," *1·2.

[12] *Ibid.*, *1·3.

[13] *Ibid.*, *1·4.

[14] *Ibid.*, *1·5.

[15] *Ibid.*, *1·6.

[16] See "Prin. Math.," *1·1.

STRICT IMPLICATION

The following set of definitions and postulates is sufficient for the system of Strict Implication.[17]

S1. $(p \supset q) = (-p \vee q)$.

S2. $(pq) = -(-p + -q)$.

S3. $(p \vee q) \supset (q \vee p)$.

S4. $(p + q) \supset (q + p)$.

S5. $(p + p) \supset p$.

S6. $q \supset (p + q)$.

S7. $(p \vee q) \supset (p + q)$.

S8. $[p \vee (q \vee r)] \supset [q \vee (p \vee r)]$.

S9. $[p \vee (q + r)] \supset [q \vee (p + r)]$.

S10. $(q \supset r) \supset [(p \vee q) \supset (p \vee r)]$.

S11. $(q \supset r) \supset [(p + q) \supset (p + r)]$.

S2 is Th. 20; S3 is Th. 13; S4 is Th. 12; S5 is Th. 21; S6 is Th. 22; S11 is Th. 30. We proceed to proof of the remainder.

Th. 37. $(p \supset q) = (-p \vee q)$. [S1]

Def. 2, $(p \supset q) = \sim (p - q)$

by P7, $(p \supset q) = \sim [-(-p) - q]$.

Def. 3, $(-p \vee q) = \sim [-(-p) - q)$

by substitution, Q. E. D.

Th. 38. $(p \vee q) \supset (p + q)$. [S7]

P9 $\{(-p-q)/p\}$: $\sim(-p-q) \supset -(-p-q)$

by Defs. 3 and 5, Q. E. D.

Th. 39. $[p \vee (q \vee r)] \supset [q \vee (p \vee r)]$ [S8]

P5 $\{-p/p;\ -q/q\}$: $[-p \supset (-q \supset r)] \supset [-q \supset (-p \supset r)]$

by Th. 37, $\{-p \supset [-(-q) \vee r]\} \supset \{-q \supset [-(-p) \vee r]\}$

by Th. 37, $\{-(-p) \vee [-(-q) \vee r]\} \supset \{-(-q) \vee [-(-p) \vee r]\}$

by P7, Q. E. D.

Th. 40. $[p \vee (q + r)] \supset [q \vee (p + r)]$. [S9]

Th. 2 $\{-q/p;\ -p/q;\ -r/r\}$:

$(\text{P4}) \supset \{\sim[-p(-q-r)] \supset \sim[-q(-p-r)]$

by P7, $\sim \{-p-[-(-q-r)]\} \supset \sim \{-q-[-(-p-r)]\}$

by Def. 3, $\{p \vee [-(-q-r)]\} \supset \{q \vee [-(-p-r)]\}$

by Def. 5, Q. E. D.

Th. 41. $(q \supset r) \supset [(p \vee q) \supset (p \vee r)]$. [S10]

P5. $(\text{P6}) \supset \{(q \supset r) \supset [(p \supset q) \supset (p \supset r)]\}$ (a);

(a) $\{-p/p\}$: $(q \supset r) \supset [(-p \supset q) \supset (-p \supset r)]$

by Th. 37, $(q \supset r) \supset \{[-(-p) \vee q] \supset [-(-p) \vee r]$

by P7, Q. E. D.

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[17] See "The Calculus of Strict Implication," *Mind*, N. S., No. 90, page 243.

Thus the entire system of strict implication is included in the matrix algebra.

Aside from its comprehensive character, the matrix algebra for implications is useful as an instrument for investigating the interrelations of necessity, truth, possibility, falsity, and impossibility, and of such derivative relations as have figured in this paper. If the assumptions of this system are true, the consequences are important not only for logic, but also for epistemology and metaphysics.

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ARE REALISM AND RELATIVITY INCOMPATIBLE?

"THE new realism," says Professor Lovejoy in a recent article in this JOURNAL,[1] "from the first had in it a strain of relativism." "Yet a thinker of the true realistic temperament," he goes on to say, "craves a world of objects which have each *some* intrinsic and solid character, which do not endlessly deliquesce into mere relations to other things, themselves equally characterless and elusive. . . . But it appears to be the fact that practically these two incongruous motives were conjoined in the production of neo-realism in its American form."

I dare say that many of us are ready to subscribe to Professor Lovejoy's contentions as here expressed. I must differ with him, however, if he holds that the realist's craving for solid *things* and the principle of relativity are inherently "two incongruous motives."

It is true that realism is prone to ignore a dependence among objects and that relativity is disposed to the opposite error of discounting an independence among them. But relativity, properly conceived, must include a specific investigation into the nature of its terms just as realism finds and has found it incumbent to recognize the effect of relations. Hence, whether for a so-called realist or for a relativist, the conjunction of things and relations appears inevitable. It is easy to perceive why a conjunction between them may stand obscured and denied; for relativity, when brought to its formulation, is as commonly lacking in its account of *things* as realism is lacking in a precise definition of its special stumbling-block,—the nature and scope of relations. Many, I have no doubt, would be surprised to find to what extent John Locke has already struggled, with no mean success, to solve this very issue. In Book III. of his Essay (to paraphrase him as nearly as possible in his

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[1] Vol. XI., page 421.